## Discrete Mathematics: Combinatorics

Background

This is a continuation of MAT2612, dealing with counting, relations, functions and partially ordered sets.

**Graph theory**

* isomorphic graphs
* planar graphs
* Euler cycles
* Hamilton circuits
* graph colouring
* trees
* the travelling salesperson problem
* minimal spanning

**Combinatorics**

* basic counting principles
* generating functions
* recurrence relations
* inclusion-exclusion principle.

**Lesson 0**

Introduction to graphs

Simple Graphs are made of two types of objects:

(no loops, multiple edges or directed edges)

Edges - Size

the number of edges in the graph

Vertices - Nodes

Order

the number of vertices in the graph

Sets in this subject follow many of the same principles as ordinary sets (subsets, improper subsets, complimentary sets etc)

A picture containing watch

Description automatically generated[1] Ordered pair with some a finite vertex set V and some Edge set E

[2] Two graphs are equal if they have the same vertex and edge sets

A picture containing watch

Description automatically generated

A picture containing watch

Description automatically generated

**Lesson 0**

Adjacency

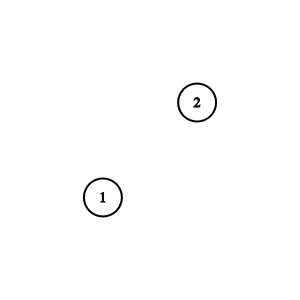
**Adjacent Vertices**

Two vertices are adjacent if they are joined by an edge

Let be a graph and , then and are adjacent in

Let be a graph, then are adjacent in there is an edge in joining a,b

Adjacent Non-adjacent

Diagram, shape, arrow

Description automatically generated

**Adjacent Edges**

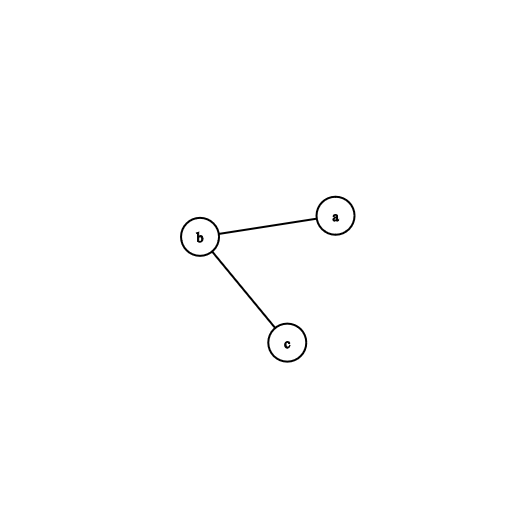
Two edges are adjacent if they are incident with a common vertex

Example

bc is incident with the vertex b

ab is incident with the vertex b

ab is adjacent to bc



Example

ab is adjacent to bc

bc is adjacent to cd

cd is adjacent to ad

ab is adjacent to bc

Shape

Description automatically generated

**Isolated Vertices**

A vertex is isolated if it not joined by any edge

**Lesson 0**

Degree of a Vertex

The degree of a vertex is the number of edges incident to it

*The number of vertices adjacent to it*

Example

a has a degree of 2

b has a degree of 1

c has a degree of 1

the minimum degree of G is 1

the maximum degree of G is 2

**Shape

Description automatically generated**

|  |  |
| --- | --- |
| **Degree of V** | **Name** |
| 0 | Isolated Vertex |
| 1 | End Vertex, Leaf |
| 2n | Even vertex |
| 2n+1 | Odd vertex |

**Theorem:** **Handshake Lemma**

The sum of degree of all vertices of a graph is twice the size of graph.

Example: ASS1 Q4

Show that a connected planar graph with at least one vertex has at least one vertex of degree at most 5

Assume that there exists a planar graph with all vertices having degree at least 6

Then:

*Where is the size of (number of edges) and*

*Where is the order of (number of vertices)*

If is planar, then we know that .

*The graph G would have at least 3 vertices*

Thus

which is a contradiction,

Thus, every planar graph has a vertex of degree at most 5

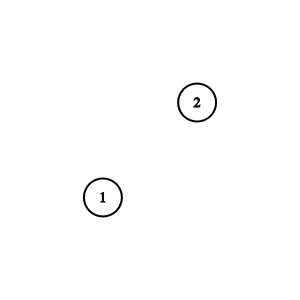
**Lesson 0**

The empty graph, trivial graph, and null graph

Example: empty graph

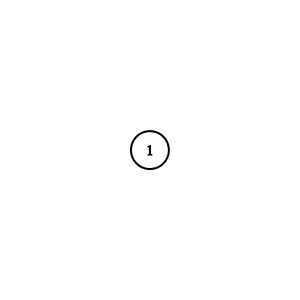
Empty Graph is any graph with some empty Edge set E

*The complement of the complete graph on 2 vertices*



Example: trivial graph

*The empty graph on 1 vertex*



Example: null graph

(Sometimes called the empty graph)

*No vertices. No edges. Is sometimes a graph, sometimes not*

**Lesson 0**

Subgraphs

* *the vertex set of H is the subset of vertex set of G AND*
* *the edge set of H is the subset of edge set of G*

Every graph is also a subgraph of itself (improper subgraph)

Example: Which of the graphs is a subgraph of another?

Diagram

Description automatically generated

Diagram

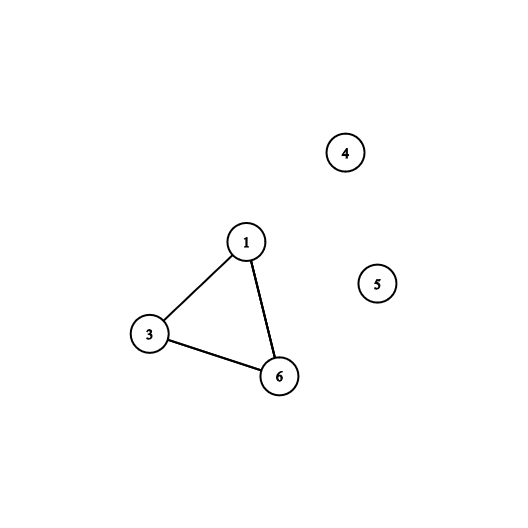
Description automatically generatedDiagram, shape

Description automatically generated

Spanning subgraph

* *the vertex set of H is equal to the subset of vertex set of G AND*
* *the edge set of H is the subset of edge set of G*

Example: Which of the graphs is a spanning subgraph of another?

Diagram

Description automatically generated

**Lesson 0**

Proper and Improper Subgraphs

Proper subset

Improper subset

**Lesson 1**

Degree sequence

This is essentially the process of listing the degrees of the vertices of a graph in a particular sequence.

Each graph has exactly one “non-increasing” degree sequence, that is, the sequence of degrees in descending order.

Let be a graph of order n.

Then, is a degree sequence of

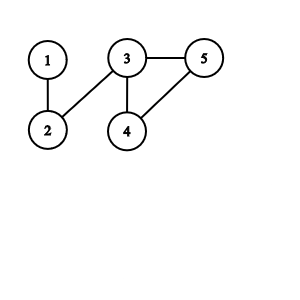
iff the vertices of can be labeled

such that for all

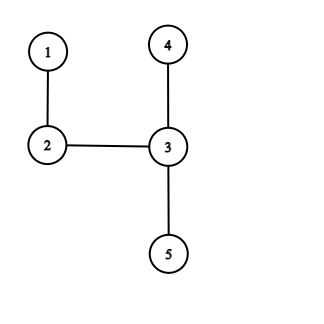
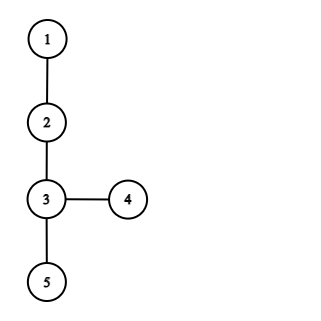
:

1,2,3,2,2

3,2,2,2,1 Non-increasing degree sequence



If two graphs are isomorphic (same structure), then their degree sequences are the same



: 3,2,1,1,1 3,2,1,1,1

But if two graphs have the same degree sequences, they are not necessarily isomorphic

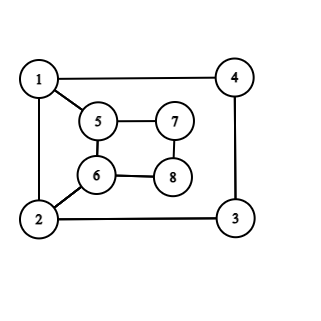
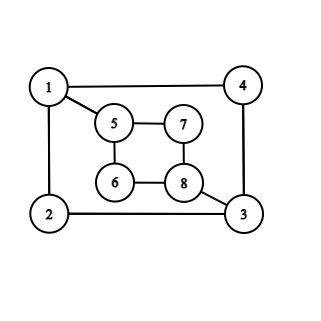
: 3,3,3,3,2,2,2,2 3,3,3,3,2,2,2,2

v5 has a degree of 3 v1 has a degree of 3

v5 has a degree of 3 v2 has a degree of 3

v5 has a degree of 3

v6 has a degree of 3



**Graphical degree sequences**

A finite sequence of non-negative integers is graphical if it is a degree sequence of some graph.

Example: some sequences which are NOT a degree sequence of some graph

3,3 *max no. of degrees in graph = no of vertices – 1*

1,1,1 *A graph must have an even number of odd vertices*

Example: ASS1 1b

Up to isomorphism, find all graphs with degree sequence (1, 1, 1, 1, 2, 2, 4)

**Isomorphism**

This is also just another way to say that two vertex sets have a one-to-one correspondence between the vertex sets.

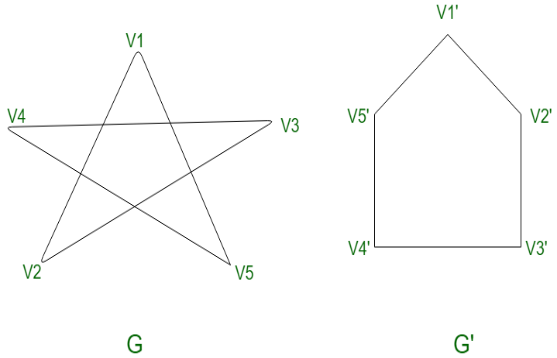
In order for two graphs to be isomorphic they should:

- equal number of nodes or

- equal number of node AND edges

- There must exist (at least) one bijective mapping from to such that two nodes are connected by an edge if and only if the corresponding nodes are connected by an edge

Example: Show that the graphs G and G’ mentioned above are isomorphic.



Let f be a bijective function from to

Let the correspondence between the graphs be-

*Match each vertex in of degree n match the corresponding vertex on*

OR

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| V1’ | V2’ | V3’ | V4’ | V5’ |
| V1 | V2 | V3 | V4 | V5 |

**Lesson 2**

Compliment of a graph

Complement of a simple graph is a simple graph having

- All the vertices of .

- An edge between two vertices and iff there exists no edge between and in the original graph .

To rephrase the above, and put together form a complete graph .

<https://math.stackexchange.com/questions/1973586/graph-theory-question-finding-the-number-of-vertices-from-number-of-edges>

**Number of Vertices**

**Number of Edges**

OR

Example: ASS1 Q1a

A graph has 10 vertices and 21 edges. Find the number of edges in the complement of .

**Number of Edges**

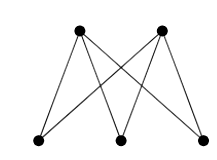
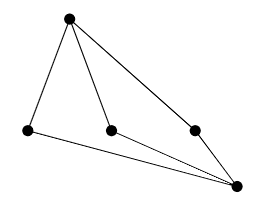
Therefore, the number of edges of the compliment of is 24

**Lesson 2**

Planar Graphs

<http://discrete.openmathbooks.org/dmoi2/sec_planar.html>

When a connected graph can be drawn without any edges crossing, it is called planar. When a planar graph is drawn in this way, it divides the plane into regions called faces.



Not planar planar – 3 faces

When a planar graph is drawn without edges crossing, the edges and vertices of the graph divide the plane into regions. We will call each region a face (including the outside region).

There is a connection between the number of vertices , the number of edges and the number of faces in any connected planar graph. This relationship is called Euler's formula.

**How to determine if a graph is planar**

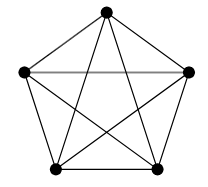
**METHOD 1: Euler's Formula for Planar Graphs**

For any (connected) planar graph with

vertices, e edges and faces, we have

Example: using Euler’s formula, prove that is not planar

*Remember is the complete graph of order 5*



– Not Planar

But does not have 7 faces. Therefore, is not planar.

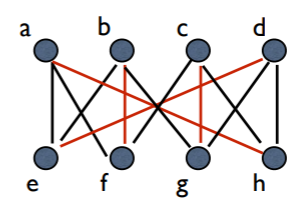
**How to determine if a graph is planar**

**METHOD 2: Circle-Chord (Hamiltonian circuit)**

<https://people.cs.umass.edu/~barring/cs575f16/lecture/4.pdf>

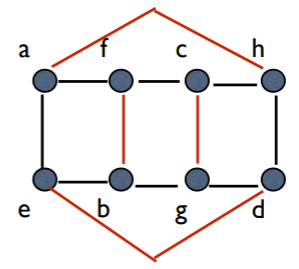
If a graph with such a circuit has a planar embedding, then it must be possible to draw the graph with that circuit as a circle.

Every other edge of the graph must then be a chord, connecting two vertices on the circle either inside it or outside

****Example 1

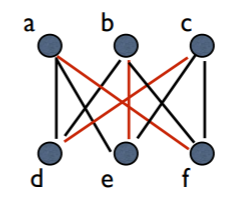
Here is a graph with 8 nodes and 12 edges. It has a Hamilton circuit

a-f-c-h-d-g-be-a, in black.

****

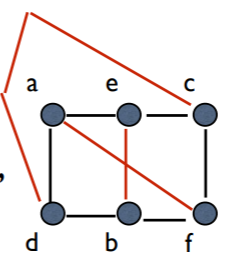
Redrawing the graph, we now have to place the four red edges. We can put bf and cg in, and ah and ed out.

Example 2

****

This graph, called has 6 nodes and 9 edges. Hamilton circuit

a-e-c-f-b-d is in black.

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But now we can’t place the three red edges without crossing. If af goes in, cd must be out, and there is no place for be either in or out.

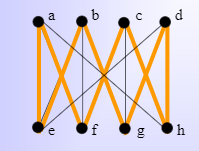
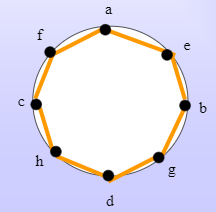
**Circle-Chord method**

<https://studylib.net/doc/9387547/section-1.4-planar-graphs>

Step 1: Find a circuit that contains all the vertices of our graph (draw it as a large circle)

Note: Finding a circuit which includes all vertices is difficult.

Recall: circuit is a path that ends where it began

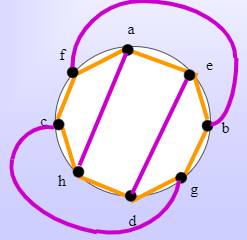
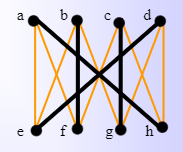


Find a circuit. The orange highlights the circuit on this graph. Draw this circuit as a circle.

Step 2: The remaining non-circuit edges, called chords, must be drawn either inside or outside the circle in a planar drawing.

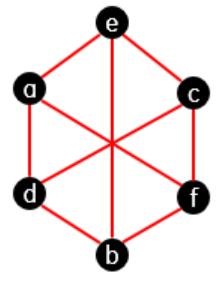
Choose a chord to draw, either inside or outside the circle. For example, we will start with the chord (b,f), drawing it outside the circle.

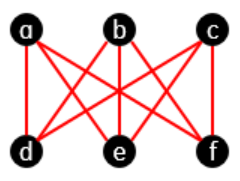
Since no lines cross another, this graph is planar!

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**Kuratowski’s Theorem**

A configuration is a where the edges have been subdivided into paths, and similarly for configurations. “Containing” means having a configuration as a subgraph.





– Not Planar – Not Planar

*is a bipartite graph with and vertices in its two vertex sets and all possible edges between vertices in the set.*

Thus, if we can find a or configuration in the graph, we know it is non-planar. In practice, most small non-planar graphs contain a configuration, and the circle-chord method is often able to find it

- basic definitions & properties of graphs

(simple mathematical models of networks)

- modelling of real-life problems in terms of graphs

- isomorphism

- special types of graphs: connected graphs

- special types of graphs: bipartite graphs