## Discrete Mathematics: Combinatorics

Background

This is a continuation of MAT2612, dealing with counting, relations, functions and partially ordered sets.

**Graph theory**

* isomorphic graphs
* planar graphs
* Euler cycles
* Hamilton circuits
* graph colouring
* trees
* the travelling salesperson problem
* minimal spanning

**Combinatorics**

* basic counting principles
* generating functions
* recurrence relations
* inclusion-exclusion principle.

**Lesson 0**

Introduction to graphs

Simple Graphs are made of two types of objects:

(no loops, multiple edges or directed edges)

Edges - Size

the number of edges in the graph

Vertices - Nodes

Order

the number of vertices in the graph

Sets in this subject follow many of the same principles as ordinary sets (subsets, improper subsets, complimentary sets etc)

A picture containing watch

Description automatically generated[1] Ordered pair with some a finite vertex set V and some Edge set E

[2] Two graphs are equal if they have the same vertex and edge sets

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**Lesson 0**

Adjacency

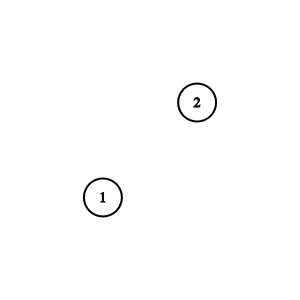
**Adjacent Vertices**

Two vertices are adjacent if they are joined by an edge

Let be a graph and , then and are adjacent in

Let be a graph, then are adjacent in there is an edge in joining a,b

Adjacent Non-adjacent

Diagram, shape, arrow

Description automatically generated

**Adjacent Edges**

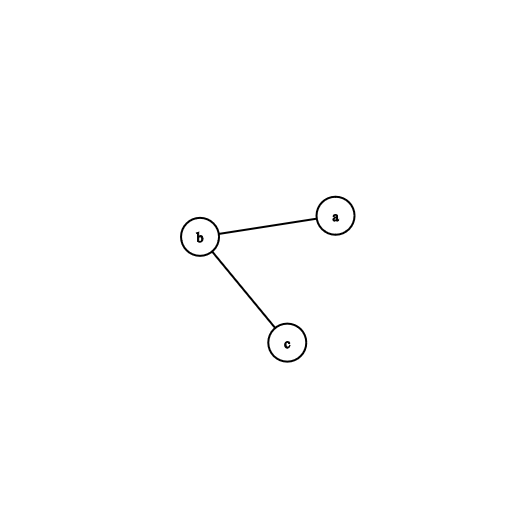
Two edges are adjacent if they are incident with a common vertex

Example

bc is incident with the vertex b

ab is incident with the vertex b

ab is adjacent to bc



Example

ab is adjacent to bc

bc is adjacent to cd

cd is adjacent to ad

ab is adjacent to bc

Shape

Description automatically generated

**Isolated Vertices**

A vertex is isolated if it not joined by any edge

**Lesson 0**

Degree of a Vertex

The degree of a vertex is the number of edges incident to it

*The number of vertices adjacent to it*

Example

a has a degree of 2

b has a degree of 1

c has a degree of 1

the minimum degree of G is 1

the maximum degree of G is 2

**Shape

Description automatically generated**

|  |  |
| --- | --- |
| **Degree of V** | **Name** |
| 0 | Isolated Vertex |
| 1 | End Vertex, Leaf |
| 2n | Even vertex |
| 2n+1 | Odd vertex |

**Theorem:** **Handshake Lemma**

The sum of degree of all vertices of a graph is twice the size of graph.

Example: ASS1 Q4

Show that a connected planar graph with at least one vertex has at least one vertex of degree at most 5

Assume that there exists a planar graph with all vertices having degree at least 6

Then:

*Where is the size of (number of edges) and*

*Where is the order of (number of vertices)*

If is planar, then we know that .

*The graph G would have at least 3 vertices*

Thus

which is a contradiction,

Thus, every planar graph has a vertex of degree at most 5

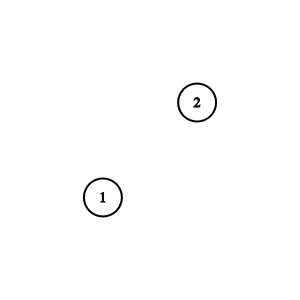
**Lesson 0**

The empty graph, trivial graph, and null graph

Example: empty graph

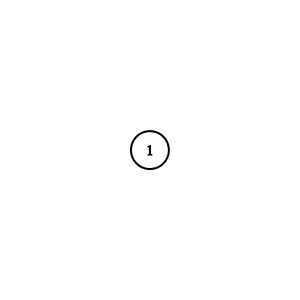
Empty Graph is any graph with some empty Edge set E

*The complement of the complete graph on 2 vertices*



Example: trivial graph

*The empty graph on 1 vertex*



Example: null graph

(Sometimes called the empty graph)

*No vertices. No edges. Is sometimes a graph, sometimes not*

**Lesson 0**

Subgraphs

* *the vertex set of H is the subset of vertex set of G AND*
* *the edge set of H is the subset of edge set of G*

Every graph is also a subgraph of itself (improper subgraph)

Example: Which of the graphs is a subgraph of another?

Diagram

Description automatically generated

Diagram

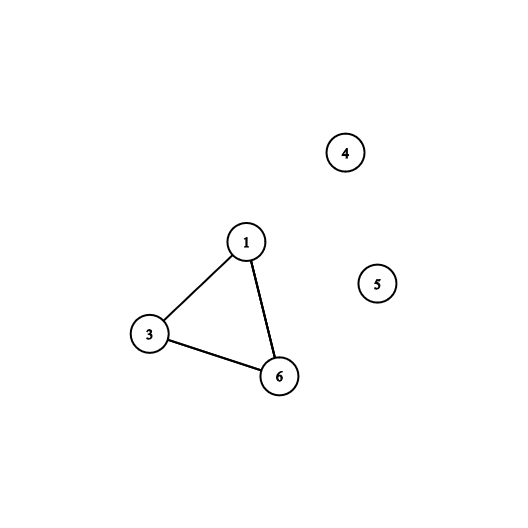
Description automatically generatedDiagram, shape

Description automatically generated

Spanning subgraph

* *the vertex set of H is equal to the subset of vertex set of G AND*
* *the edge set of H is the subset of edge set of G*

Example: Which of the graphs is a spanning subgraph of another?

Diagram

Description automatically generated

**Lesson 0**

Proper and Improper Subgraphs

Proper subset

Improper subset

**Lesson 1**

Degree sequence

This is essentially the process of listing the degrees of the vertices of a graph in a particular sequence.

Each graph has exactly one “non-increasing” degree sequence, that is, the sequence of degrees in descending order.

Let be a graph of order n.

Then, is a degree sequence of

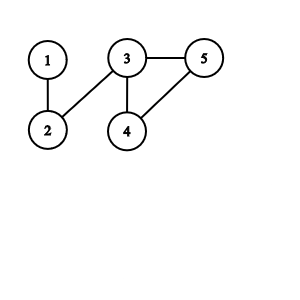
iff the vertices of can be labeled

such that for all

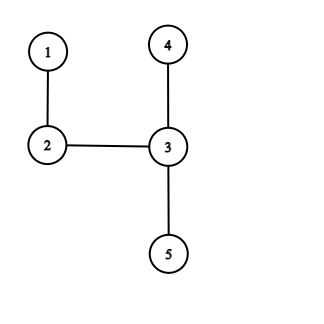
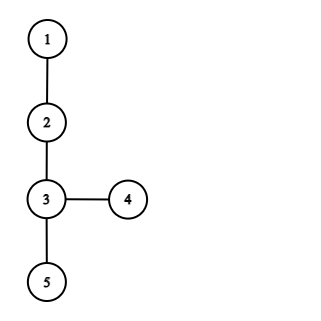
:

1,2,3,2,2

3,2,2,2,1 Non-increasing degree sequence



If two graphs are isomorphic (same structure), then their degree sequences are the same



: 3,2,1,1,1 3,2,1,1,1

But if two graphs have the same degree sequences, they are not necessarily isomorphic

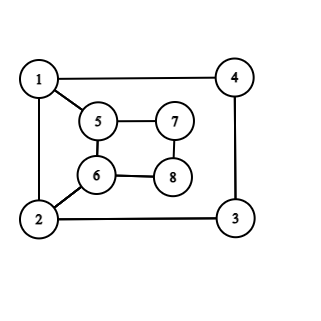
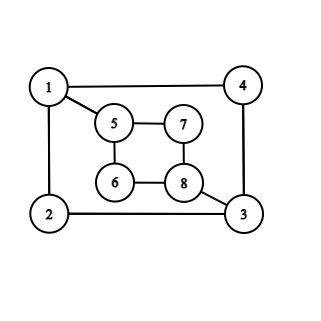
: 3,3,3,3,2,2,2,2 3,3,3,3,2,2,2,2

v5 has a degree of 3 v1 has a degree of 3

v5 has a degree of 3 v2 has a degree of 3

v5 has a degree of 3

v6 has a degree of 3



**Graphical degree sequences**

A finite sequence of non-negative integers is graphical if it is a degree sequence of some graph.

Example: some sequences which are NOT a degree sequence of some graph

3,3 *max no. of degrees in graph = no of vertices – 1*

1,1,1 *A graph must have an even number of odd vertices*

Example: ASS1 1b

Up to isomorphism, find all graphs with degree sequence (1, 1, 1, 1, 2, 2, 4)

**Isomorphism**

*NB: There's no known efficient algorithm that is guaranteed to tell you whether two graphs are isomorphic. Of course, we could try all possible permutations of the vertices, but this will take a very long time. We know heuristics: good things to try which will work in many cases, but will sometimes give us an inconclusive answer.*

This is also just another way to say that two vertex sets have a one-to-one correspondence between the vertex sets.

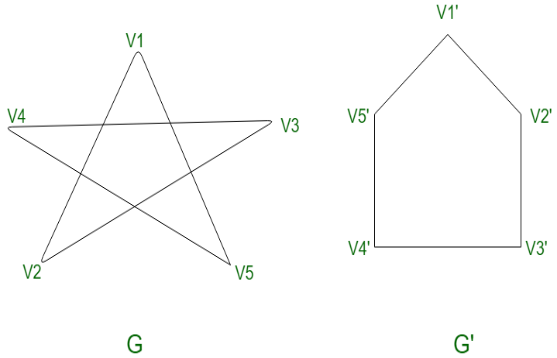
In order for two graphs to be isomorphic they should:

- equal number of nodes or

- equal number of node AND edges

- There must exist (at least) one bijective mapping from to such that two nodes are connected by an edge if and only if the corresponding nodes are connected by an edge

Example: Show that the graphs G and G’ mentioned above are isomorphic.



Let f be a bijective function from to

Let the correspondence between the graphs be-

*Match each vertex in of degree n match the corresponding vertex on*

OR

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| V1’ | V2’ | V3’ | V4’ | V5’ |
| V1 | V2 | V3 | V4 | V5 |

**Steps to determine isomorphism (of two or more graphs)**

[0] Basic check: check if graph has equal number of nodes AND equal number of edges

[1] Find degree sequences of all the graphs

[2] Count the number of 3,4 & 5 circuits to other graphs

**Example: NOV 2015**

Determine whether the following two graphs are isomorphic or not.

If they are, give a vertex correspondence.

If they are not, give a reason why.

**Chart, radar chart

Description automatically generated**

[1] Find degree sequences of all the graphs

**Graph**

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Edge** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** |
| **Degree** | **3** | **3** | **3** | **3** | **4** | **3** | **4** | **3** |

**degree sequence is (4,4,3,3,3,3,3,3)**

**Graph**

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Edge** | **a** | **b** | **c** | **d** | **e** | **f** | **g** | **h** |
| **Degree** | **3** | **4** | **3** | **3** | **3** | **3** | **4** | **3** |

**degree sequence is (4,4,3,3,3,3,3,3)**

[2] The first graph has four 3-circuits

(1−5−6−1, 5−6−7−5, 5−7−4−5 and 2−3−8−2)

The second graph has three 3-circuits

(a−e−f −a, g−b−h−g, h−b−c−h)

**Steps to determine isomorphism (of two or more graphs)**

[1] Find degree sequences of all the graphs

[2] Count the number of 3 circuits to other graphs

[3] Count the number of 4 circuits to other graphs

[3] Count the number of 5 circuits to other graphs

**Example: NOV 2015 Q1c**

Determine whether the following two graphs are isomorphic or not.

If they are, give a vertex correspondence.

If they are not, give a reason why.

**Chart, radar chart

Description automatically generated**

[1] Find degree sequences of all the graphs

**Graph**

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Edge** | **a** | **b** | **c** | **d** | **e** | **f** | **g** |  |
| **Degree** | **2** | **3** | **2** | **3** | **2** | **4** | **2** |  |

**degree sequence is (4,3,3,2,2,2,2)**

**Graph**

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Edge** | **1** | **2** | **3** | **4** | **5** | **6** | **7** |  |
| **Degree** | **2** | **3** | **3** | **2** | **2** | **4** | **2** |  |

**degree sequence is (4,3,3,2,2,2,2)**

[2] Count the number of 3 circuits to other graphs

The first graph has four 3-circuits

(1−5−6−1, 5−6−7−5, 5−7−4−5 and 2−3−8−2)

The second graph has three 3-circuits

(a−e−f −a, g−b−h−g, h−b−c−h)

**Lesson 2**

Compliment of a graph

Complement of a simple graph is a simple graph having

- All the vertices of .

- An edge between two vertices and iff there exists no edge between and in the original graph .

To rephrase the above, and put together form a complete graph .

**Formal definition**

Let be a graph of order

*Order* *: the number of vertices in the graph*

The complement graph is the graph with

*has the same vertex set*

and

*has the same edge set*

**Number of Vertices**

**Number of Edges**

OR

List of complete graphs from to

Background pattern

Description automatically generated with medium confidence

<https://math.stackexchange.com/questions/1973586/graph-theory-question-finding-the-number-of-vertices-from-number-of-edges>

Example: ASS1 Q1a

A graph has 10 vertices and 21 edges. Find the number of edges in the complement of .

**Number of Edges**

Therefore, the number of edges of the compliment of is 24

Example NOV 2104

If a graph has 41 edges, what is the smallest number of vertices it can have?

**Number of Edges**

Number of vertices

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|  | 6 | 12 | 20 | 30 | 42 | 56 | 72 | 90 | 110 |

Thus we see that .

Before we can now say that , we have to make sure there is a graph with vertices and edges.

But this is easy: just remove any 4 edges from

(which has edges).

**Example**

Draw the complement of the graph

**A picture containing diagram

Description automatically generated**

**Number of vertices will remain the same**

**Number of Edges**

Therefore, the number of edges of the compliment of is 6

**Diagram

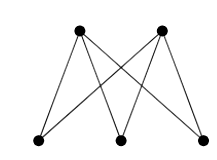
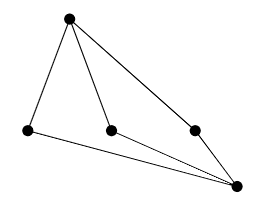
Description automatically generated**

**Lesson 2**

Planar Graphs

<http://discrete.openmathbooks.org/dmoi2/sec_planar.html>

When a connected graph can be drawn without any edges crossing, it is called planar. When a planar graph is drawn in this way, it divides the plane into regions called faces.



Not planar planar – 3 faces

When a planar graph is drawn without edges crossing, the edges and vertices of the graph divide the plane into regions. We will call each region a face (including the outside region).

There is a connection between the number of vertices , the number of edges and the number of faces in any connected planar graph. This relationship is called r's formula.

**How to determine if a graph is planar**

**Method 1: quick algorithm check**

For a simple, connected, planar graph with v vertices and e edges and f faces, the following simple conditions hold for v ≥ 3:

Theorem 1:

;

*planetary criteria*

Theorem 2:

If there are no cycles of length 3, then .

*planetary criteria*

Theorem 3:

.

**How to determine if a graph is planar**

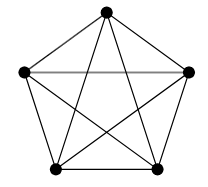
**Method 2: Euler's Formula for Planar Graphs**

For any (connected) planar graph with

vertices, e edges and faces, we have

Example: using Euler’s formula, prove that is not planar

*Remember is the complete graph of order 5*



– Not Planar

But does not have 7 faces. Therefore, is not planar.

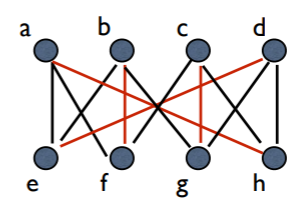
**How to determine if a graph is planar**

**Method 3: Circle-Chord (Hamiltonian circuit)**

<https://people.cs.umass.edu/~barring/cs575f16/lecture/4.pdf>

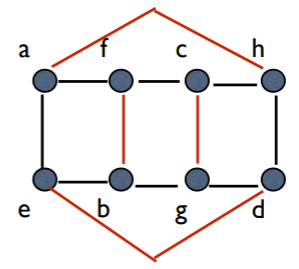
If a graph with such a circuit has a planar embedding, then it must be possible to draw the graph with that circuit as a circle.

Every other edge of the graph must then be a chord, connecting two vertices on the circle either inside it or outside

****Example 1

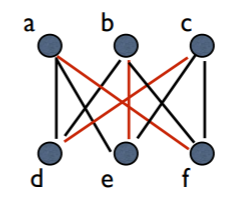
Here is a graph with 8 nodes and 12 edges. It has a Hamilton circuit

a-f-c-h-d-g-be-a, in black.

****

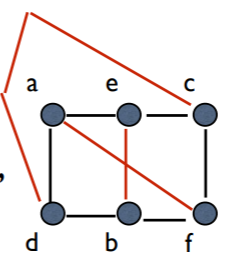
Redrawing the graph, we now have to place the four red edges. We can put bf and cg in, and ah and ed out.

Example 2

****

This graph, called has 6 nodes and 9 edges. Hamilton circuit

a-e-c-f-b-d is in black.

****

But now we can’t place the three red edges without crossing. If af goes in, cd must be out, and there is no place for be either in or out.

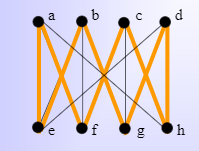
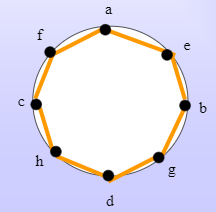
**Circle-Chord method**

<https://studylib.net/doc/9387547/section-1.4-planar-graphs>

Step 1: Find a circuit that contains all the vertices of our graph (draw it as a large circle)

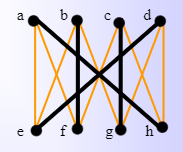
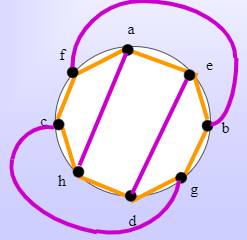
Note: Finding a circuit which includes all vertices is difficult.

Recall: circuit is a path that ends where it began



Find a circuit. The orange highlights the circuit on this graph. Draw this circuit as a circle.

Step 2: The remaining non-circuit edges, called chords, must be drawn either inside or outside the circle in a planar drawing.

****

Choose a chord to draw, either inside or outside the circle. For example, we will start with the chord (b,f), drawing it outside the circle.

Since no lines cross another, this graph is planar!

**Example: pg51 Alan Tucker Applied Combinatorics**

**Shape, polygon

Description automatically generated**

**Method 2: try using squares (8 vertices)**

[Step 1] Draw vertices (8 vertices)

[Step 2] Label vertices. And draw edges

Vertices that cross one/two edges should be the neighbours

*a-e and a-f*

*b-e and b-f (since b has similar connections as a)*

Vertices that have three or more should be further out

*a-h*

*b-g*

*c-g*

*c-h*

*d-g*

*d-h*

*attempt 1: close but there is no connection between c-e.*

*we can resolve this by swapping e and f*

a

e

c

h

f

b

g

d

Missing c-f

Missing d-e (easy to make with an outside edge)

Missing a-h (easy to make with an outside edge)

*attempt 2: outside edges can be made easily*

a

f

c

h

e

b

g

d

Example: Nov 2015 Q4

**A picture containing chart

Description automatically generated**

[Step 2] Label vertices. And draw edges

Vertices that cross one/two edges should be the neighbours

*1-5 and 1-2 and 1-4*

*2-6 and 2-3 and 2-1*

*3-6 and 3-6 and 3-2*

Vertices that have three or more should be further out

*1-8*

*2-7*

*3-7*

*attempt 1: swap 4 and 5*

1

2

3

8

5

4

6

7

Missing: 1-8

Missing: 2-7

Missing: 2-4

Missing: 3-4

Missing: 4-8

Missing: 5-6

Missing: 6-8

Missing: 5-7

*attempt 2: swap 3 and 6, 7 and 8*

1

2

3

7

4

5

6

8

Missing: 1-8

Missing: 3-4

Missing: 5-7

Cannot draw 1-8, 3-4 and 5-7

configuration

6

7

1

8

5

2

Example: NOV2014 Q5

Chart, radar chart

Description automatically generated

Theorem 1:

;

*planetary criteria*

;

;

Which holds true

Theorem 2:

If there are no cycles of length 3, then .

*planetary criteria*

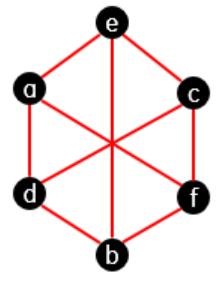
Theorem 3:

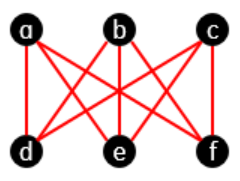
.

Which holds true

**Kuratowski’s Theorem**

A configuration is a where the edges have been subdivided into paths, and similarly for configurations. “Containing” means having a configuration as a subgraph.





– Not Planar – Not Planar

*is a bipartite graph with and vertices in its two vertex sets and all possible edges between vertices in the set.*

Thus, if we can find a or configuration in the graph, we know it is non-planar. In practice, most small non-planar graphs contain a configuration, and the circle-chord method is often able to find it

**Lesson 0**

Circuits and Paths

**Hamiltonian path – visits every vertex once with no repeats but does not have to start and end at the same vertex**

*A picture containing colorful, clock, wire, bright

Description automatically generated*

**Hamiltonian circuit – visits every vertex once with no repeats.**

**It must end at the same vertex**

**A picture containing clock

Description automatically generated**

**Euler path – uses every edge in a graph with no repeats.**

**Does not return to the starting vertex**

*You can cross the same vertices if necessary*

A graph will contain an Euler path if it contains at most two vertices of an odd degree

*Diagram, schematic

Description automatically generated*

**Euler circuit – uses every edge in a graph with no repeats.**

**Must start and end at the same vertex**

A graph will contain an Euler circuit if all vertices have an even degree

**Diagram

Description automatically generated**

**Lesson 0**

Proper Colouring – Colouring the Graph G

This is an assignment of colours to the vertices of G such that adjacent vertices are coloured differently.

3 vertices

3 colours

3 vertices

2 colours

Both of the above graphs are valid

**Chromatic number of G**

The smallest number of colours of G is called the chromatic number of G, denoted .

A graph G with chromatic number is called K-chromatic

Example

<https://www.youtube.com/watch?v=3VeQhNF5-rE>

Shape, polygon

Description automatically generatedRadar chart

Description automatically generated

K-Colouring

6-Colouring

Minimum Colouring

4-chromatic

10 vertices so:

K colourable for

For our purposes, instead of colours we can just use positive integers that represent different colours

A graph is said to be triangle free if no two adjacent vertices are

adjacent to a common vertex.

Example: NOV 2015

A picture containing boat

Description automatically generated

Determine the chromatic number of .

Give a minimal colouring and a proof that a smaller number of colours is not possible

[1] Find any defining characteristics (minimum number of colours)

Since the graph contains triangles, χ(G) ≥ 3.

[2] Next we try to colour the graph with the minimum number of colours (three colours).

A picture containing boat

Description automatically generated

For the triangle abga we colour:

a – blue

g – red

b - green.

Then since c is adjacent to g and b:

c - blue.

Next we note that for the triangle aefa:

e - red

f - green.

But then vertex d will be adjacent to red, green and blue vertex. We can therefore not colour the vertices with only three colours.

Hence .

|  |  |  |  |
| --- | --- | --- | --- |
| Blue | Red | Green | Yellow |
| a,c,j | e,g,i | b,d,f | h |

**Lesson 0**

Bipartite Graphs

A graph whose vertices can be divided into two independent sets and

such that there is no edge that connect vertices of the same set

A bipartite graph is possible:

- If the graph colouring is possible using two colours ( and )

Such that vertices in a set are coloured with the same colour

Theorem 2: A graph G is bipartite if and only if every circuit in G is of even length

Some examples:

Diagram

Description automatically generated

Example: NOV 2015 Q5b

Chart, radar chart

Description automatically generated

Determine whether or not is bipartite

***Theoreom 2 of Tucker: A graph is bipartite if and only if every circuit in is of even length.***

It is not bipartite because it contains odd circuits, e.g. abga.

For which values of and does the complete bipartite graph, have an Euler cycle?

**Solution.**

Let and be the two partite sets of .

Then every vertex has degree and every vertex has degree .

Since a graph has an Euler cycle if and only if all the vertices are even,

will have an Euler cycle if and only if and are both even.

Are there any ’s that have Euler trails but not Euler cycles? Explain.

**Solution.**

A graph has an Euler trail but not an Euler cycle if and only if it has exactly two vertices of odd degree.

Hence will have an Euler trail but not an Euler cycle if and n is odd.

Diagram, engineering drawing

Description automatically generated

**Lesson 0**

Peterson Graphs

**P5 Graph**

Shape

Description automatically generated with medium confidence

**P6 Graph**

Blue lights in the dark

Description automatically generated with low confidence

**P7 Graph**

Chart

Description automatically generated

- basic definitions & properties of graphs

(simple mathematical models of networks)

- modelling of real-life problems in terms of graphs

- isomorphism

- special types of graphs: connected graphs

- special types of graphs: bipartite graphs

Example: NOV 2015

Use the circle-chord method to determine whether the graph below is planar. If it is, give a planar drawing. If it is not, find a or configuration.

Step 1: Find a circuit that contains all the vertices of our graph Hamilton circuit: 1-2-3-6-5-7-8-4

A picture containing diagram

Description automatically generated

1

2

3

6

5

7

8

4

Step 2: The remaining non-circuit edges, called chords, must be drawn either inside or outside the circle in a planar drawing

A picture containing diagram

Description automatically generated

1

2

3

6

5

7

8

4

2-7, 5-1, 5-4 all cannot be drawn.

Therefore, the graph is not planar.

configuration

A picture containing diagram

Description automatically generated

1

5

7

2

6

8

**Lesson 0**

Counting: Permutations and Combinations

*From MAT2612*

**[SELECT] Permutation**:

number of elements in the set

number of elements chosen

* An ordered arrangement of unlike objects.
* An r-permutation is the arrangement of r-elements of a set

no repetition:

fixed repetition:

unlimited repetition:

Example: Let . Find all 2-permutations

Example: How many ways can 100 marathon runners place 1st, 2nd, and 3rd?

**[ARRANGE] Combination**:

number of elements in the set

number of elements chosen

* An unordered arrangement of unlike objects.
* An r-combination is a subset of the set with r-elements

Example: Let . Find all 2-combinations. Relate this to the number of 2-permutations

Example: how many poker hands of 5 cards can be dealt from a standard of 52 cards

Lesson 0

Counting:

*From MAT2612*

**Product Rule (AND)**

* Procedure can be broken down into a sequence of tasks
* Number of outcomes of each task is
* different ways to perform a procedure

Example: If I have 4 different t-shirts and 3 different pairs of shorts, how many outfits do I have?

**Sum Rule (OR)**

* If a task can be performed in one of or ways, then there are ways to perform the task.

Example: I want to take a trip to the beach. I can travel to one of 37 international beaches or one of 14 domestic beaches. How many beach destination choices do I have?

A close up of a logo

Description automatically generated**Subtraction Rule (Inclusion exclusion)**

* If a task can be done in either one of or ways, then the total number of ways to do the task is minus the number of ways that are common

Example: how many bitstrings of length 7 either start with 1 bit or end with 3 bits 000?

Number that start [1][ ][ ][ ][ ][ ][ ]

with 1

Number that end [ ][ ][ ][ ][0][0][0]

with 000

Number that start with 1 [1][ ][ ][ ][0][0][0]

and end in 000

**Division Rule**

* There are is ways to do a task if it can be done using a procedure that can be carried out in ways, where the corresponding outcomes per group.

Example: how many ways can I sit 6 people around a circular table where two seats are considered the same when each person has the same left and right neighbour?

***Set one person and then the rest***

Lesson 0

Counting: Multinomial Theory (The Mississippi Problem)

<https://medium.com/i-math/can-you-solve-the-mississippi-problem-6c0f3b02531>

[1] How many distinguishable arrangements are there of the letters in the word MISSISSIPPI?

MISSISSIPPI

**Order doesn’t matter**

Elements

Repetition:

[2] How many distinguishable arrangements are there of the letters in the word MISSISSIPPI?

**No adjacent P’s**

MISSISSII

**Order doesn’t matter**

Elements

Repetition: *P’s are removed*

\_M\_I\_S\_S\_I\_S\_S\_I\_I\_

**Order matters**

Elements *spaces to insert P’s*

[2] How many distinguishable arrangements are there of the letters in the word MISSISSIPPI?

**Adjacent S’s: INCLUSION-EXCLUSION PRINCIPLE**

Permutations(MISSISSIPII) – No Adjacent S’s

MIIIPPI

**Order doesn’t matter**

Elements

Repetition: *S’s are removed*

\_M\_I\_I\_P\_P\_I\_I\_

**Order matters**

Elements *spaces to insert S’s*

Example: NOV2015

Lesson 0

Distributions

Distributions of distinct objects are equivalent to arrangements

number of elements in the set

number of elements chosen

**[ARRANGE] Combination**:

* An unordered arrangement of unlike objects.
* An r-combination is a subset of the set with r-elements

Distributions of identical objects are equivalent to selections

**[SELECT] Permutation**:

number of elements in the set

number of elements chosen

* An ordered arrangement of unlike objects.
* An r-permutation is the arrangement of r-elements of a set

no repetition:

fixed repetition:

unlimited repetition:

Equivalent Forms for Selection with Repetition

1. The number of ways to select objects with repetition from n different types of objects.

2. The number of ways to distribute identical objects into n distinct boxes.

3. The number of nonnegative integer solutions to

Example: Nov2015 Q6

Integer solutions

Find the number of integer solutions to

[1] convert inequality into equation

let

[2] rewrite constraints (each constraint represents a container)

*5 containers in total*

[3] model problem

We can place 50 objects into 5 distinct containers

Container 1 will have at least 6 objects

Container 2 will have at least 6 objects

Container 3 will have at least 8 objects

Container 4 will have at least 8 objects

[4] use model to solve

number of elements in the set

number of elements chosen

objects will need to be placed